AN EXCURSION THROUGH FIBONACCI NUMBER CURIOS

Consider the infinite simple continued fraction in which every element equals unity. The successive continuants of this fraction are

1, 2/1, 3/2, 5/3, 8/5, 13/8, 21/13, ..(1)

and one cannot fail to notice a clear pattern in the above sequence. The numerator of each of these elements equals the denominator in the next element. Further, it is evident that the sum of the numerator and denominator of each fraction in the infinite chain equals the numerator of the next element. It is also true that the sequence has to converge to an irrational number, since it is well known that every infinite simple continued fraction necessarily converges to an irrational number. The sequence of numbers.

(f): 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ..(2)

is intimately connected to the sequence (1) which is seen to be the set of ratios of successive elements of (2). Alternatively, the sequence of positive integers (f) in (2) is just the set of denominators of the successive fractions in (1). The sequence (f) is the set of Fibonacci numbers, named after the 12th century Italian mathematician Leonardo Fibonacci. The sequence (f) has the obvious property that every element of the sequence beyond the second in the chain equals the sum of the two immediate predecessor elements. This recurrence law coupled with the initialization that the first two elements are both equal to

unity, completely specifies the Fibonacci sequence (f_n). This sequence occurs in almost every branch of mathematics like number theory, differential equations, probability theory, statistics, numerical analysis, linear algebra and also other areas like biology, chemistry and electrical engineering. It is easy to show that the general (i.e. n-th) element of the chain (f) equals

$$f_n = \{(1+\sqrt{5})^n - (1-\sqrt{5})^n\}/(2^n)\sqrt{5} \qquad ..(3)$$

The sequence (f) has close affinity with the binominal coefficients $\binom{n}{r}$. If these coefficients are arranged in a Pascal triangular pattern with the apex element $\binom{1}{1} = 1$ at the top, it is seen that the sum of the entries along the North-East-Eastern (N.E.E) rising diagonal, add up to the elements of the chain (f). This may be put in the form that the figure of the binomial coefficients viewed through an angle of $\pi/8$ from the horizontal, yields the chain of Fibonacci numbers.

The sequence of ratios (1) above converges to the irrational number

$$\phi = \frac{1+\sqrt{5}}{2} \qquad 1.618034 \qquad ..(4)$$

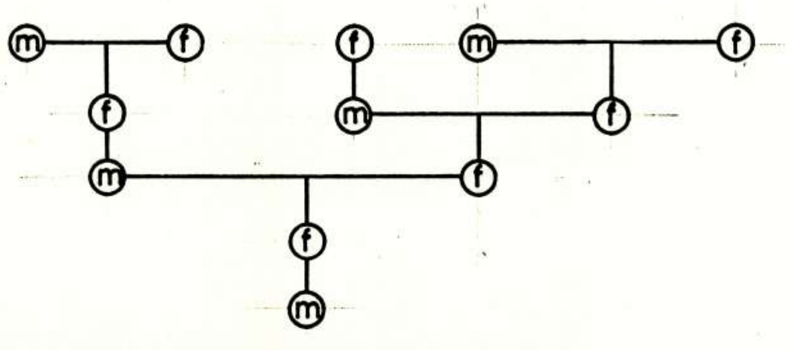
and this number has a deep geometrical significance. It corresponds to the point on a given line sequent that divides the segment in the golden ratio (ϕ) mentioned above and this geometric construction shows that it is possible to construct geometrically (i.e. with ruler and compasses only) the angle $\pi/10$ and also an isosceles triangle in which the vertical angle equals

half the base angle.

The golden section ratio ϕ has the property : $\phi = (\phi-1)^{-1}$.

Consider a rectangle (edges a, b with b > a) of aspect ratio $b/a = \phi$. It a square on the shorter edge a is scooped out of the rectangle, the remaining figure is a rectangle with the longer side = a and shorter side $b-a = a(\phi-1)$ and the aspect ratio of this rectangle is also ϕ, so that this rectangle is similar to the original rectangle with orientation phase lag = $-\pi/2$. The process is repetitive in nature and psychologists are of the view that of all the rectangles, those with the aspect ratio close to ϕ are the most shapely figures.

Nature seems to abound in Fibonacci number patterns. A drone (the male species of the bee) has a mother but no father. Eggs laid by a (female) bee and fertilized, hatch into females while male bees (drones) hatch from unfertilized eggs. The geneaological tree of a male bee is easily traced backwards and it is seen that in any one generation, the number of ancestors is a Fibonacci number.



(5)
(3)
(2)
(1)
(1)

This is an instance in nature where simple arithmetic can be displayed with precision. Discussion on the development of a

plant on the growth of a weed on a pond or the descendants of a bee involves an element of variation on a regular pattern of development, owing to changes and chances of natural life. Nature abounds in variety but with an underlying uniformity. However, it is no simple task to discern the underlying uniformity.

Each species of flower has a basic pattern of development and a corresponding number - pattern typical of the species. The pattern may not precisely be followed in every single plant of the species. Plants with symmetrical development and four-petalled flowers, follow other patterns. The sequence of Fibonacci numbers plays a very significant part in the description of flowers.

Members of the lily family have <u>three</u> petals. There are hundreds of species of flowers with <u>five</u> petals. <u>Eight</u> petals are not so common, but there are quite a few such species. <u>Thirteen</u>, <u>twenty one</u> and <u>thirty four</u> petals are also known to exist. <u>Fifty five</u> and <u>eighty nine</u> petals are common with Michaelmas diasies.

The association of Fibonacci numbers and plants is not restricted to details of flowers only. A new shoot of a plant commonly grows out at the axil, the point where a leaf springs from the main stem of the plant. Further leaves and branch shoots grow from both the main and branch stems. The number of leaves or nodes where new shoots appear in the next stage of development, equals the sum of those from the main and branch stems. This development has similarity with the structure of the

genealogical tree of the bee. It is no doubt, difficult to see an actual plant so perfectly displaying the Fibonacci sequence. The effect of sunlight only on one side and blowing of wind from another direction, occasional cold spels and various other factors do retard growth of plants at several stages of development and there are thus departures from the general pattern of species. However, there is no gain saying that normally the shoots and flower heads will approximate to Fibonacci patterns.

The general Fibonacci number (f_n) is seen in (3) above and undoubtedly its forbidding appearance is one main factor for not pursuing the study of the sequence at an early stage. The golden ratio $\phi = (1+\sqrt{5})/(2)$ and the ratio $1/\sqrt{5}$ make the calculations on f_n somewhat disagreeable. However, a perusal of the table below in which

$$u_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n \text{ is approximated}$$

into

$$u_n = (0.4412736)\ (1.618034)^n \qquad ..(5)$$

and f_n = the n-th Fibonacci number shows the closeness of u_n and f_n.

The points $(n,\ u_n)$ are well approximated also by the exponential curve

$$y = (0.4472)\ (1.61803)^x \qquad ..(6)$$

The Fibonacci numbers (f_n) are a set of positive integers

n	u_n	f_n
1	0.7236	1
2	1.1708	1
3	1.8944	2
4	3.0652	3
5	4.9597	5
6	8.0249	8
7	12.9846	13
8	21.0095	21
9	33.9941	34
10	55.0036	55
11	88.9977	89
12	144.0014	144
13	232.9991	233

which satisfy an exponential law of growth, almost exactly. Though counting of leaves, branches, flowers and bees always yields whole numbers, there is an unmistakable relationship to the exponential law of growth, which is typical of biological systems. It shows the compatibility of growth in continuous and discrete quantities.

The human ear has a specialized physiological structure that enables different sounds and combinations of sounds that are heard with discrimination. What we call music (vocal or instrumental) gives pleasure when it reaches the brain through the mechanism of the human ear. The conventions developed over the centuries in the tuning of musical instruments are aimed to

produce music that is acceptable to the ear and cannot therefore be completely arbitrary. They must in some way be compatible with the physiology of the hearing mechanism. There is a clear relationship between the exponential sequence (such as that in (6) above which defines an equiangular spiral) and the sensitive hair-like receptors graded in length to receive the whole range of resonant frequencies arranged compactly in the spiral cochlea in the (human) ears. The Fibonacci sequence has closeness to points on an exponential curve and thus there is a forceful suggestion of Fibbonaccian development about our musical conventions. Theories of harmony and musical form have to reckon with the physiological structure of the ear (related to the exponential sequences) and the individual notes that are standardized according to musical conventions over the centuries. The essential point to note is that these conventions have features that are characteristic of the Fibonacci sequence. Whether this is a mere coincidence or otherwise, had better be left as a questions open for study in greater depth.

In the last three decades the technical literature on Fibonacci numbers has vastly grown both in expanse and content. It is now a highly specialized area with several journals exclusively catering to the research conducted over several countries. It is also the subject matter of several international symposia. However, the interest generated by the study of these numbers seems to be perennial.